

Bounds for the extremal parameter of nonlinear eigenvalue problems and application to the explosion problem in a flow

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Abstract

We consider the nonlinear eigenvalue problem $Lu = \lambda f(u)$, posed in a smooth bounded domain $\Omega \subseteq \mathbb{R}^N$ with Dirichlet boundary condition, where L is a uniformly elliptic second-order linear differential operator, $\lambda > 0$ and $f : [0, a_f) \rightarrow \mathbb{R}_+$ ($0 < a_f \leq \infty$) is a smooth, increasing and convex nonlinearity such that $f(0) > 0$ and which blows up at a_f . First we present some upper and lower bounds for the extremal parameter λ^* and the extremal solution u^* . Then we apply the results to the operator $L_A = -\Delta + Ac(x)$ with $A > 0$ and $c(x)$ is a divergence-free flow in Ω . We show that, if $\psi_{A,\Omega}$ is the maximum of the solution $\psi_A(x)$ of the equation $L_A u = 1$ in Ω with Dirichlet boundary condition, then for any incompressible flow $c(x)$ we have, $\psi_{A,\Omega} \rightarrow 0$ as $A \rightarrow \infty$ if and only if $c(x)$ has no non-zero first integrals in $H_0^1(\Omega)$. Also, taking $c(x) = -x\rho(|x|)$ where ρ is a smooth real function on $[0, 1]$ then $c(x)$ is never divergence-free in unit ball $B \subset \mathbb{R}^N$, but our results completely determine the behaviour of the extremal parameter λ_A^* as $A \rightarrow \infty$.

Key words: semilinear elliptic problem, nonlinear eigenvalue problem, extremal solution.

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1. Introduction and main results

The explosion problem in a flow concerns existence and regularity of positive solutions of nonlinear eigenvalue problem of the form

$$\begin{cases} -\Delta u + c(x) \cdot \nabla u = \lambda f(u) & x \in \Omega, \\ u = 0 & x \in \partial\Omega, \end{cases} \quad (1.1)$$

where Ω is a bounded smooth domain in \mathbb{R}^N ($N \geq 2$), $\lambda > 0$, $f : [0, a_f) \rightarrow \mathbb{R}_+$ is a smooth, increasing, convex function such that $f(0) > 0$, $\int_0^{a_f} \frac{ds}{f(s)} < \infty$ which blows up at the endpoint of its domain. We consider two cases either f is a *regular nonlinearity* i.e., $D_f := [0, +\infty)$ and f is superlinear, namely $f(t)/t \rightarrow \infty$ as $t \rightarrow \infty$, or when $D_f := [0, 1)$ and $\lim_{t \nearrow 1} f(t) = +\infty$ called a *singular nonlinearity*. Typical examples of regular nonlinearities f are e^u , $(1+u)^p$ for $p > 1$, while singular nonlinearities include $(1-u)^{-p}$ for $p > 1$.

It is said that a solution of problem (1.1) is classical provided $\|u\|_{L^\infty} < \infty$ (resp., $\|u\|_{L^\infty} < 1$) if f is a regular (resp., singular) nonlinearity. It is known that there exists an extremal parameter (critical threshold) $\lambda^* \in (0, \infty)$ depending on Ω , $c(x)$ and N , such that problem (1.1) has a unique minimal classical solution $u_\lambda \in C^2(\overline{\Omega})$ if $0 < \lambda < \lambda^*$ while no solution exists, even in the weak sense, for $\lambda > \lambda^*$. One can show that $\lambda \mapsto u_\lambda(x)$ is increasing in λ for all $x \in \Omega$ and therefore one can define the extremal solution $u^*(x) = \lim_{\lambda \nearrow \lambda^*} u_\lambda(x)$, which is a weak solution of problem (1.1) at $\lambda = \lambda^*$. The regularity of solutions at $\lambda = \lambda^*$ is a delicate issue. In the case that endpoint of the domain f is finite, Cowan and Ghoussoub in [11]

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proved that the extremal solution of problem (1.1) with $f(u) = \frac{1}{(1-u)^2}$ is regular for all $1 \leq N \leq 7$. Luo, Ye and Zhou in [15] proved that the extremal solution is regular in the low-dimensional case. In particular, for the radial case, all extremal solutions are regular in dimension two. When $c \equiv 0$, the regularity of u^* has been studied extensively in the literature [2, 7, 8, 11, 12, 16, 19]. For example, we know that when $f(u) = e^u$ or $f(u) = (1+u)^p$, then u^* is regular in dimensions $N \leq 9$. For general nonlinearities f , Nedev [16] proved the regularity of u^* in dimensions $N = 2, 3$. In dimension $N = 4$ the same is proved by Cabré [9] when Ω is convex (without assuming the convexity of f), and by Villegas [18] for arbitrary domains and f is convex. However, it is still an open problem to establish the regularity of u^* in dimensions $5 \leq N \leq 9$ for regular nonlinearities f . Ghoussoub and Guo in [14] showed that when Ω is a ball and $f(u) = \frac{1}{(1-u)^2}$, then u^* is singular if $N \geq 8$, while it is regular if $N < 8$.

In this work, first we consider semilinear second-order elliptic equation of the form

$$\begin{cases} Lu = \lambda f(u) & x \in \Omega, \\ u = 0 & x \in \partial\Omega, \end{cases} \quad (1.2)$$

where L is a second-order linear differential operator acting on functions $u : \Omega \rightarrow \mathbb{R}$ which is uniformly elliptic and has the following nondivergence general form

$$Lu = - \sum_{i,j=1}^N a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + c(x) \cdot \nabla u,$$

where $c(x) = (c_1(x), c_2(x), \dots, c_n(x))$ is a smooth vector field on $\overline{\Omega}$ and $a_{i,j}(x) = a_{j,i}(x)$ are smooth functions. The linear operator L can be also showed in the divergence form as

$$Lu = - \sum_{i,j=1}^N \frac{\partial}{\partial x_j} \left(a_{i,j}(x) \frac{\partial u}{\partial x_i} \right) + b(x) \cdot \nabla u,$$

where $b(x) = (b_1(x), b_2(x), \dots, b_N(x))$ and $b_i(x) = c_i(x) + \sum_{j=1}^N \frac{\partial a_{i,j}(x)}{\partial x_j}$ for all $1 \leq i \leq N$. When the linear operator L has divergence form the linear operator L^* , the formal adjoint of L , is

$$L^*u = - \sum_{i,j=1}^N \frac{\partial}{\partial x_i} \left(a_{i,j}(x) \frac{\partial u}{\partial x_j} \right) - \operatorname{div}(u(x)b(x)).$$

Fredholm alternative theorem and regularity theory imply that the following equation

$$\begin{cases} Lu = 1 & x \in \Omega, \\ u = 0 & x \in \partial\Omega, \end{cases}$$

has a unique nonnegative smooth solution [13]. This solution will be denoted by ψ_L and will be called the torsion function for uniformly elliptic operator L . If $L = -\Delta$, then we omit L and just write ψ . We shall denote $\psi_{L,\Omega} := \sup_{x \in \Omega} \psi_L(x) = \|\psi_L\|_\infty$ and $\psi_\Omega := \sup_{x \in \Omega} \psi(x) = \|\psi\|_\infty$. We also denote by $(\eta(x), \mu_1(L^*, \Omega))$, the first eigenpair of adjoint problem

$$\begin{cases} L^*\eta = \mu_1(L^*, \Omega)\eta & x \in \Omega, \\ \eta = 0 & x \in \partial\Omega, \end{cases} \quad (1.3)$$

A nonnegative solution $u_\lambda(x)$ of (1.2) is said to be minimal if for any other solution v of (1.2) we have $u_\lambda(x) \leq v(x)$ for all $x \in \Omega$. Also, we say that a solution $v(x)$ of (1.2) is stable if the principal eigenvalue κ_1 of the linearized operator $\tilde{L}_\lambda \varphi = L\varphi - \lambda f'(v)\varphi$ is positive.

Fix a flow profile $c(x)$ and consider the following problem

$$\begin{cases} -\Delta u + Ac(x) \cdot \nabla u = \lambda f(u) & x \in \Omega, \\ u = 0 & x \in \partial\Omega, \end{cases} \quad (1.4)$$

where A is a positive number. Denote by $\lambda^*(A)$, ψ_A and $\psi_{A,\Omega}$, the extremal parameter of problem (1.4), the torsion function for the linear operator $L_A = -\Delta + Ac(x) \cdot \nabla$ and $\psi_{A,\Omega} = \sup_{x \in \Omega} \psi_A(x)$, respectively.

H. Berestycki and collaborators [5], by using the ideas from [6, 3, 10], showed that in problem (1.4) when $c(x)$ is divergence-free (incompressible) i.e., $\operatorname{div} c(x) = 0$, then

Theorem A. We have $\lambda^*(A) \rightarrow \infty$ as $A \rightarrow \infty$ if and only if u has no non-zero first integrals in $H_0^1(\Omega)$.

Recall that a function $\Psi \in H^1(\Omega)$ is a first integral of u if $u \cdot \nabla \Psi = 0$ a.e. in Ω . They also proved that $\psi_{A,\Omega} \rightarrow 0$ as $A \rightarrow \infty$ if $c(x)$ has no first integrals in $H_0^1(\Omega)$ (see Lemma 3.2 in [5]). Indeed, the proof of their result based on the key observation that one can write $\psi_A(x) = \int_0^\infty \xi(t, x) dt$ where the function $\xi(t, x)$ solves a special parabolic problem on $[0, \infty) \times \Omega$ discussed in [17]. In this paper, we prove the condition that $\psi_{A,\Omega} \rightarrow 0$ as $A \rightarrow \infty$ is also sufficient (see the following theorem) and we give a rather simple proof for the necessary condition using only the maximum principle.

Theorem 1.1. *For any incompressible flow $c(x)$ in problem (1.4) we have $\psi_{A,\Omega} \rightarrow 0$ as $A \rightarrow \infty$, if and only if $c(x)$ has no non-zero first integrals in $H_0^1(\Omega)$.*

Another illustration of how our results are applicable, we consider semilinear second-order elliptic equations of the form

$$\begin{cases} -\Delta u - A\rho(|x|)x \cdot \nabla u = \lambda f(u) & x \in B, \\ u = 0 & x \in \partial B, \end{cases} \quad (1.5)$$

where $B := B(0, 1)$, $A \geq 0$, $\lambda > 0$, $\rho : [0, 1] \rightarrow \mathbb{R}$ is a smooth function and $c(x) := -x\rho(|x|)$, $x \in B$ is a smooth vector field and $f : [0, a_f) \rightarrow \mathbb{R}_+$ is regular or singular nonlinearity. Notice that $c(x)$ is never divergence-free as $\operatorname{div} c(x) = 0$ implies that $\rho(|x|) = \frac{a}{|x|^N}$ ($x \neq 0$) for some constant a which is impossible, because ρ is assumed to be continuous on $[0, 1]$.

The following theorem, completely determine the behavior of extremal parameter of problem (1.5).

Theorem 1.2. *Consider problem (1.5), then*

- (i) *If there exists $x_0 \in [0, 1]$ such that $\rho(x_0) < 0$, then $\psi_{L_A, B} \rightarrow \infty$ as $A \rightarrow \infty$. This implies that for all nonlinearities f we have $\lambda^*(A) \rightarrow 0$ as $A \rightarrow \infty$.*
- (ii) *If $\rho \geq 0$ and $\rho \not\equiv 0$ on any interval $I \subseteq [0, 1]$, then $\psi_{L_A, B} \rightarrow 0$ as $A \rightarrow \infty$. This implies that for all nonlinearities f we have $\lambda^*(A) \rightarrow \infty$ as $A \rightarrow \infty$.*
- (iii) *If $\rho \geq 0$ and $\rho \equiv 0$ on some interval $I \subseteq [0, 1]$, then there exists positive constant $C_{N, \rho}$ where $C_{N, \rho}$ depends on ρ, N and independent of A such that*

$$C_{N, \rho} \leq \psi_{L_A, B} \leq \frac{1}{2N} \quad \text{for all } A \geq 0.$$

Consequently, for all nonlinearities f there exist positive constants $D_{N, f}, \tilde{D}_{N, \rho, f}$ where $D_{N, f}$ depends on N and f but not A and $\tilde{D}_{N, \rho, f}$ depends on ρ, N and f but not A such that

$$D_{N, f} \leq \lambda^*(A) \leq \tilde{D}_{N, \rho, f} \quad \text{for all } A \geq 0.$$

The authors in [5] also proved that the critical threshold λ^* for (1.1) when $c(x)$ is incompressible cannot close to zero, precisely, for any domain Ω and regular nonlinearity f there exists $\lambda_0 > 0$ so that the extremal parameter λ^* of problem (1.1) satisfies $\lambda^* \geq \lambda_0 > 0$ for all incompressible flows $c(x)$ in Ω . The constant λ_0 depends on Ω and the function f . They also showed that this result does not hold without the restriction that the flow $c(x)$ is incompressible and give an example $c_n(x) = 4nx$ for all $n \in \mathbb{N}$ such that $c_n(x)$ is never divergence-free and the critical threshold for (1.1) tends to zero as n tends to infinity. To show this in [5] (in dimension two and $\Omega = B$), by setting $\Psi_n = e^{-n|x|^2} \Theta_n$ where Θ_n is a radial solution of problem (1.4) with

$c_n(x) = 4nx$ for some λ_n , they obtained a self-adjoint problem for Θ_n , then using suitable test function in the variational principle for λ_n proved that $\lambda_n \leq Ce^{-cn} \rightarrow 0$ as $n \rightarrow \infty$ which implies that $\lambda_n^* \rightarrow 0$ as well. This result, however, is a direct consequence of our Theorem 1.2 part (i) by taking $\rho(|x|) = -4$.

In this paper, before proving Theorems 1.1 and 1.2, we consider the general semilinear eigenvalue problem (1.2) and shall present some sharp upper and lower bounds for the extremal parameter for the general nonlinearity f (regular or singular) as well as pointwise lower and upper bounds on the minimal stable solution u_λ of (1.2). Our first proposition establishes the existence as well as lower and upper bounds of the extremal parameter of problem (1.2).

Proposition 1.1. *There exists $\lambda^*(L, \Omega, f) \in (0, \infty)$ such that:*

- (i) *for every $0 < \lambda < \lambda^*(L, \Omega, f)$ the problem (1.2) has a unique positive classical solution $u_\lambda(x)$ which is minimal and stable. Furthermore, this extremal parameter satisfies*

$$\frac{1}{\psi_{L,\Omega}} \sup_{0 < t < a_f} \frac{t}{f(t)} \leq \lambda^*(L, f, \Omega) \leq \mu_1(L^*, \Omega) \sup_{0 < t < a_f} \frac{t}{f(t)}. \quad (1.6)$$

- (ii) *for each $x \in \Omega$, the function $\lambda \mapsto u_\lambda(x)$ is differentiable and strictly increasing on $(0, \lambda^*)$.*

- (iii) *there exists no classical solution of (1.2) for $\lambda > \lambda^*(L, \Omega, f)$.*

The proof of this result is very close to that in [5], but for the convenience of the reader we present it in this paper. In the following theorem, we give another upper bound for the extremal parameter of problem (1.2) which, in many cases, represent a sharper upper bound than (1.6). We also give pointwise lower bound for the extremal solution of problem (1.2). Throughout this paper, for all nonlinearity $f : [0, a_f] \rightarrow \mathbb{R}_+$, we define the function $F : [0, a_f] \rightarrow \mathbb{R}_+$ as follows

$$F(t) = \int_0^t \frac{ds}{f(s)}. \quad (1.7)$$

Theorem 1.3. *Let $u \in C^2(\Omega)$ be a solution of problem (1.2), then*

$$F^{-1}(\lambda \psi_L(x)) \leq u(x) \quad \text{for all } x \in \Omega,$$

where F is defined in (1.7). Therefore if $x_0 \in \Omega$ such that $\psi_L(x_0) = \psi_{L,\Omega}$, then

$$\lambda \leq \frac{F(u(x_0))}{\psi_{L,\Omega}}.$$

In particular, we have

$$\lambda^* \leq \frac{F(u^*(x_0))}{\psi_{L,\Omega}} \leq \frac{F(a_f)}{\psi_{L,\Omega}} \quad \text{and} \quad F^{-1}(\lambda^* \psi_L(x)) \leq u^*(x) \quad \text{for all } x \in \Omega. \quad (1.8)$$

To see the sharpness of above results, consider the following problem

$$\begin{cases} Lu = \lambda f(u^p) & x \in \Omega, \\ u = 0 & x \in \partial\Omega, \end{cases} \quad (1.9)$$

where $p \geq 1$ and $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is an increasing, convex and superlinear C^2 -function such that $f(0) > 0$. In the following theorem, we show that upper bound (1.8) for the extremal parameter of problem (1.9) is arbitrarily close to lower bound (1.6) provided that p is sufficiently large. This also implies that upper bound (1.8) is an improvement of (1.6).

Theorem 1.4. *Consider semilinear second-order elliptic equation (1.9). Then*

$$\lim_{p \rightarrow \infty} \lambda_p^* = \frac{1}{f(0)\psi_{L,\Omega}} \quad \text{and} \quad \lim_{p \rightarrow \infty} \|u_p^*\|_\infty = +\infty,$$

where λ_p^* and u_p^* are the extremal parameter and extremal solution (respectively) of problem (1.9).

In the following theorem, we give another lower bound for the extremal parameter of problem (1.2) which is a better lower bound, at least when $L = -\Delta$, than (1.6) for more values of N . We also give pointwise upper bound for the minimal solution of problem (1.2) for all $\lambda \in (0, \bar{\lambda})$ where $\bar{\lambda} \leq \lambda^*$ is given in below.

Theorem 1.5. *Consider the semilinear elliptic equation (1.2), then*

$$\lambda^*(L, f, \Omega) \geq \sup_{0 < \alpha < \frac{\|F\|_\infty}{\psi_{L,\Omega}}} \alpha - \alpha^2 \beta(\alpha) := \bar{\lambda}, \quad (1.10)$$

where $\beta(\alpha) := \sup_{x \in \Omega} f'(F^{-1}(\alpha\psi_L(x))) |\nabla \psi_L(x)|^2$ and F is defined in (1.7). Furthermore, if we define $\lambda(\alpha) = \alpha - \alpha^2 \beta(\alpha)$ for all $0 \leq \alpha \leq \|F\|_\infty / \psi_{L,\Omega}$, then

$$u_{\lambda(\alpha)}(x) \leq F^{-1}(\alpha\psi_L(x)) \quad \text{for all } 0 \leq \alpha \leq \frac{\|F\|_\infty}{\psi_{L,\Omega}}.$$

The authors in [1] show that lower bound (1.10) gives the exact value of the extremal parameter λ^* when $L = -\Delta$, $f(u) = e^u$, $f(u) = (1+u)^p$ and $f(u) = (1-u)^{-p}$ in some dimensions.

Using the above theorems we get

Proposition 1.2. *Assume that u_λ is the minimal solution of problem (1.2) and F is defined in (1.7), then*

(i) *for each $x \in \Omega$, the function $\lambda \mapsto \frac{F(u_\lambda(x))}{\lambda}$ is increasing on $(0, \lambda^*)$. In particular,*

$$u_\lambda(x) \leq F^{-1}\left(\frac{\lambda}{\lambda^*} \|F\|_\infty\right) \quad \text{for all } \lambda \in (0, \lambda^*).$$

(ii) $\frac{F(u_\lambda(x))}{\lambda} \rightarrow \psi_L$ uniformly as $\lambda \rightarrow 0^+$.

Note that the first assertion of Proposition 1.2 gives an upper bound for the minimal solution of problem (1.2) which is an interesting issue in itself. For example, consider the following problem

$$\begin{cases} -\Delta u = \lambda e^u & x \in \Omega, \\ u = 0 & x \in \partial\Omega. \end{cases}$$

Here we have $f(t) = e^t$, $F(t) = 1 - e^{-t}$, $\|F\|_\infty = 1$ and $F^{-1}(t) = \ln \frac{1}{1-t}$. Taking $\lambda^* = \lambda^*(e^t, \Omega)$, then, by part (i) of Proposition 1.2 we have

$$u_\lambda(x) \leq \ln \frac{\lambda^*}{\lambda^* - \lambda} \quad \text{for all } \lambda \in (0, \lambda^*).$$

If $N > 9$ and $\Omega = B(0, 1)$, then $\lambda^*(e^t, B(0, 1)) = 2N - 4$ [7], so we have

$$u_\lambda(x) \leq \ln \frac{2N - 4}{2N - 4 - \lambda} \quad \text{for all } \lambda \in (0, 2N - 4).$$

2. Existence and basic properties of the extremal parameter

In this section, we prove Proposition 1.1 which is well known when $L = -\Delta$, and also prove the first assertion of Proposition 1.2. To do these, first we give a nonexistence result for the nonlinear eigenvalue problem (1.2).

Lemma 2.1. *The problem (1.2) admits no classical solutions for $\lambda > \mu_1(L^*, \Omega) \sup_{0 < t < a_f} \frac{t}{f(t)}$.*

Proof. Clearly,

$$\int_{\Omega} (L^* \eta - \mu_1(L^*, \Omega) \eta) u \, dx = 0,$$

for any solution u of (1.2). Now, integration by parts implies that

$$\int_{\Omega} \eta (\lambda f(u) - u \mu_1(L^*, \Omega)) dx = 0,$$

and thus there exists $x \in \Omega$ such that $\lambda f(u(x)) - u(x) \mu_1(L^*, \Omega) < 0$. It follows that

$$\lambda \leq \mu_1(L^*, \Omega) \sup_{0 < t < a_f} \frac{t}{f(t)}.$$

This completes the proof. \square

Now, we show that there exists a constant $C > 0$ such that for all $\lambda \in (0, C)$ the problem (1.2) has a positive classical solution.

Lemma 2.2. *Problem (1.2) admits a minimal nonnegative solution $u_{\lambda}(x)$ for all $\lambda \leq \frac{1}{\psi_{L, \Omega}} \sup_{0 < t < a_f} \frac{t}{f(t)}$.*

To prove Lemma 2.2, we construct a super-solution and using it we show that a positive solution of (1.2) exists. To do that, we need the following well-known fact.

Lemma 2.3. *Suppose that there exists a smooth function $\bar{u}(x)$ satisfying*

$$\begin{cases} L\bar{u} \geq \lambda f(\bar{u}) & x \in \Omega, \\ \bar{u} \geq 0 & x \in \partial\Omega. \end{cases} \quad (2.1)$$

Then there exists a classical solution u_{λ} of (1.2) which is minimal.

Proof. Let $u_0 \equiv 0$ and define an approximating sequence $u_n(x)$ such that $u_{n+1}(x)$ is the smooth solution of

$$\begin{cases} Lu_{n+1} = \lambda f(u_n) & x \in \Omega, \\ u_{n+1} = 0 & x \in \partial\Omega. \end{cases}$$

From the maximum principle we know that $0 \leq u_0 \leq \bar{u}$. Now by induction, assuming $0 \leq u_{n-1} \leq \bar{u}$ for some $n \in \mathbb{N}$, we get

$$\begin{cases} L(\bar{u} - u_n) \geq \lambda[f(\bar{u}) - f(u_{n-1})] \geq 0 & x \in \Omega, \\ \bar{u} - u_n \geq 0 & x \in \partial\Omega, \end{cases}$$

concludes that $0 \leq u_n \leq \bar{u}$. In a similar way, the maximum principle implies that the sequence $\{u_n\}$ is monotone increasing. Therefore, the sequence $\{u_n\}$ converges uniformly to a limit u_{λ} which has to be a classical solution of (1.2) and satisfies $0 \leq u_{\lambda} \leq \bar{u}$. Since this inequality holds for any solution of (2.1), then u_{λ} is a minimal positive solution of (1.2) and is clearly unique. \square

Proof of Lemma 2.2. Choose $\alpha > 0$ such that

$$\frac{\alpha\psi_{L,\Omega}}{f(\alpha\psi_{L,\Omega})} = \sup_{0 < t < a_f} \frac{t}{f(t)},$$

and consider the smooth function $\bar{u}(x) = \alpha\psi_L(x)$ for $x \in \Omega$. Clearly, we have

$$\begin{cases} L\bar{u} = \alpha \geq \lambda f(\bar{u}) & x \in \Omega, \\ \bar{u} = 0 & x \in \partial\Omega, \end{cases}$$

provided that $\lambda \leq \frac{\alpha}{f(\alpha\psi_{L,\Omega})} = \frac{1}{\psi_{L,\Omega}} \frac{\alpha\psi_{L,\Omega}}{f(\alpha\psi_{L,\Omega})} = \frac{1}{\psi_{L,\Omega}} \sup_{0 < t < a_f} \frac{t}{f(t)}$. Now, existence of a minimal solution to (1.2) follows from Lemma 2.3. \square

The following two lemmas show that any minimal solution of (1.2) is stable. We recall that for any minimal solution u_λ of (1.2) we denote by $\kappa_1(\lambda, u_\lambda)$ the principal eigenvalue corresponding to positive eigenfunction ϕ of the following linearized operator \tilde{L}_λ

$$\tilde{L}_\lambda \varphi = L\varphi - \lambda f'(u_\lambda)\varphi \quad \text{for all } \varphi \in C^2(\Omega) \quad (2.2)$$

Lemma 2.4. *For any minimal solution of (1.2) we have $\kappa_1(\lambda, u_\lambda) \geq 0$.*

Proof. Assume that u_λ is a minimal solution of (1.2) and the principal eigenvalue $\kappa_1(\lambda, u_\lambda)$ of the problem

$$\begin{cases} L\phi - \lambda f'(u_\lambda)\phi = \kappa_1(\lambda, u_\lambda)\phi & x \in \Omega, \\ \phi = 0 & x \in \partial\Omega, \end{cases}$$

is negative. Consider the function $\phi_\epsilon = u_\lambda - \epsilon\phi$, then we have

$$\begin{aligned} L\phi_\epsilon - \lambda f(\phi_\epsilon) &= \lambda f(u_\lambda) - \epsilon\lambda f'(u_\lambda)\phi - \epsilon\kappa_1(\lambda, u_\lambda)\phi - \lambda f(u_\lambda - \epsilon\phi) \\ &= -\epsilon\kappa_1(\lambda, u_\lambda)\phi + \lambda(f(u_\lambda) - \epsilon f'(u_\lambda)\phi - f(u_\lambda - \epsilon\phi)) \\ &= -\epsilon\kappa_1(\lambda, u_\lambda) - \frac{\epsilon^2 f''(\xi)}{2}\phi^2 \geq 0, \end{aligned}$$

provided that ϵ is sufficiently small. This means that problem (1.2) has a classical solution, say u , which satisfies $u \leq \phi_\epsilon < u_\lambda$ by Lemma (2.3). This contradicts the minimality of u_λ . So, we have $\kappa_1(\lambda, u_\lambda) \geq 0$ if u_λ is a minimal solution. \square

Lemma 2.5. *Let u_λ be a solution of (1.2) such that $\kappa_1(\lambda, u_\lambda) = 0$. Then no classical solution of (1.2) with $\bar{\lambda} > \lambda$ exists.*

Proof. We argue by contradiction. Suppose that $\bar{\lambda} > \lambda$ and there exists a function $\bar{u} \geq 0$ such that

$$\begin{cases} L\bar{u} = \bar{\lambda}f(\bar{u}) & x \in \Omega, \\ \bar{u} = 0 & x \in \partial\Omega. \end{cases}$$

Also, denote by ϕ the positive eigenfunction of the adjoint problem

$$\begin{cases} L^*\phi = \lambda f'(u_\lambda)\phi & x \in \Omega, \\ \phi = 0 & x \in \partial\Omega. \end{cases} \quad (2.3)$$

Set $\eta_\tau = u_\lambda + \tau(\bar{u} - u_\lambda)$ for all $\tau \in [0, 1]$. Then convexity of f implies that

$$\begin{aligned} L\eta_\tau - \lambda f(\eta_\tau) &= L\eta_\tau - \lambda f(\tau\bar{u} + (1-\tau)u_\lambda) \\ &\geq L\eta_\tau - \lambda\tau f(\bar{u}) - \lambda(1-\tau)f(u_\lambda) \\ &= \tau f(\bar{u})(\bar{\lambda} - \lambda) \geq 0, \end{aligned} \quad (2.4)$$

for all $\tau \in [0, 1]$. Moreover, $L\eta_0 = \lambda f(\eta_0)$. If we differentiate (2.4) with respect to τ at $\tau = 0$, then we have the following inequality for $\xi = \bar{u} - u_\lambda$:

$$L\xi - \lambda f'(u_\lambda)\xi \geq (\bar{\lambda} - \lambda)f(\bar{u}) > 0. \quad (2.5)$$

Multiplying (2.5) by the eigenfunction ϕ of (2.3) and integrating by part, one obtains

$$0 < \int_{\Omega} \phi(L\xi - \lambda f'(u_\lambda)\xi)dx = \int_{\Omega} \xi(L^*\phi - \lambda f'(u_\lambda)\phi)dx = 0,$$

which is a contradiction. Therefore, there exists no classical solution of (1.2) for $\bar{\lambda} > \lambda$ if $\kappa_1(\lambda, u_\lambda) = 0$. \square

Notice that the above lemma also proves that the extremal parameter of problem (1.2) can be determined by

$$\lambda^*(L, \Omega, f) = \sup \left\{ \lambda > 0 : \text{the minimal solution } u_\lambda \text{ of problem (1.2) is stable} \right\}.$$

The following lemma completes the proof of Proposition 1.1.

Lemma 2.6. *Let u_λ be the minimal solution of (1.2) for $\lambda \in (0, \lambda^*)$, then for each $x \in \Omega$ the function $\lambda \mapsto u_\lambda(x)$ is strictly increasing and differentiable on $(0, \lambda^*)$.*

Proof. Suppose that $0 < \lambda_1 < \lambda_2 < \lambda^*$, then clearly we have $Lu_{\lambda_1} = \lambda_1 f(u_{\lambda_1}) \leq \lambda_1 f(u_{\lambda_2}) = \frac{\lambda_1}{\lambda_2} \lambda_2 f(u_{\lambda_2}) = \frac{\lambda_1}{\lambda_2} Lu_{\lambda_2}$. This means that

$$\begin{cases} L(u_{\lambda_1} - \frac{\lambda_1}{\lambda_2} u_{\lambda_2}) \leq 0 & x \in \Omega, \\ u_{\lambda_1} - \frac{\lambda_1}{\lambda_2} u_{\lambda_2} = 0 & x \in \partial\Omega. \end{cases}$$

Now, maximum principle implies that $\frac{u_{\lambda_1}}{\lambda_1} \leq \frac{u_{\lambda_2}}{\lambda_2}$. It follows that $u_{\lambda_1} < u_{\lambda_2}$.

Fix $\lambda_0 \in (0, \lambda^*)$ and define the operator P such that $P(\lambda, \Phi) = L\Phi - \lambda f(\Phi)$ for all $\lambda \in (0, \lambda^*)$ and $\Phi \in C^2(\Omega) \cap C(\partial\Omega)$ such that $\Phi = 0$ on $\partial\Omega$. Clearly, P is a C^1 map and $P(\lambda_0, u_{\lambda_0}) = 0$. On the other hand $d_\Phi P(\lambda_0, u_{\lambda_0}) = \tilde{L}_{\lambda_0}$, where \tilde{L}_{λ_0} is defined by (2.2) and $d_\Phi P(\lambda_0, u_{\lambda_0})$ is derivative of the function $p(\lambda, \Phi)$ with respect to Φ . Since u_{λ_0} is stable, the linearized operator \tilde{L}_{λ_0} is invertible. By the Implicit Function Theorem, $\lambda \mapsto u_\lambda(x)$ is differentiable at λ_0 and by monotonicity, $\frac{du_\lambda}{d\lambda}(x) \geq 0$ for all $x \in \Omega$. \square

In the following, we prove the first assertion of Proposition 1.2.

Proof of Proposition 1.2. (i). Let $0 < \lambda_1 < \lambda_2 < \lambda^*$ be arbitrary and set $\alpha = \frac{\lambda_1}{\lambda_2}$. Consider the function $\bar{u}(x) = F^{-1}(\alpha F(u_{\lambda_2}(x)))$ for all $x \in \Omega$. Note that since $\alpha < 1$ and the function f' is increasing, then $f'(u_{\lambda_2}) - \alpha f'(\bar{u}) > 0$. Letting $A := [a_{i,j}(x)]_{i,j}$ which is a symmetric matrix and positive definite for all

$x \in \Omega$, then it can be easily checked that

$$\begin{aligned}
L\bar{u} &= - \sum_{i,j=1}^N a_{i,j} \frac{\partial^2 \bar{u}}{\partial x_i \partial x_j} + c(x) \cdot \nabla \bar{u} \\
&= - \sum_{i,j=1}^N a_{i,j} \left(\frac{\alpha^2 f'(\bar{u}) - \alpha f'(u_{\lambda_2})}{f^2(u_{\lambda_2})} \frac{\partial u_{\lambda_2}}{\partial x_i} \frac{\partial u_{\lambda_2}}{\partial x_j} + \frac{\alpha}{f(u_{\lambda_2})} \frac{\partial^2 u_{\lambda_2}}{\partial x_i \partial x_j} \right) f(\bar{u}) \\
&\quad + \frac{\alpha f(\bar{u})}{f(u_{\lambda_2})} c(x) \cdot \nabla u_{\lambda_2} \\
&= - \sum_{i,j=1}^N a_{i,j} \left(\frac{\alpha^2 f'(\bar{u}) - \alpha f'(u_{\lambda_2})}{f^2(u_{\lambda_2})} \frac{\partial u_{\lambda_2}}{\partial x_i} \frac{\partial u_{\lambda_2}}{\partial x_j} \right) f(\bar{u}) + \frac{\alpha f(\bar{u})}{f(u_{\lambda_2})} L u_{\lambda_2} \\
&= \left(\alpha \nabla u_{\lambda_2} A (\nabla u_{\lambda_2})^T \frac{f'(u_{\lambda_2}) - \alpha f'(\bar{u})}{f^2(u_{\lambda_2})} + \lambda_1 \right) f(\bar{u}) \geq \lambda_1 f(\bar{u}).
\end{aligned}$$

It then follows that $\bar{u}(x)$ is a super-solution of

$$\begin{cases} Lu = \lambda_1 f(u) & x \in \Omega, \\ u = 0 & x \in \partial\Omega, \end{cases}$$

Hence, by Lemma 2.3, we have $u_{\lambda_1} \leq \bar{u}$, so $\frac{F(u_{\lambda_1})}{\lambda_1} \leq \frac{F(u_{\lambda_2})}{\lambda_2}$. \square

Uniform L^∞ -bounds for the functions u_λ at $\lambda = \lambda^*$ are difficult to obtain. In the following, we prove that when we are away from λ^* a uniform L^∞ -bound exists which is not depend on the domain Ω and the linear operator L .

Theorem 2.1. *For any $0 < \delta < 1$ we have*

$$0 \leq u_\lambda(x) \leq C(\delta, f) := F^{-1}\left((1-\delta)\|F\|_\infty\right) \text{ for all } 0 < \lambda \leq (1-\delta)\lambda^*.$$

Note that $C(\delta, f)$ depends only on δ and nonlinearity $f(t)$ but not on the domain Ω or the linear operator L .

Proof. Fix $0 < \delta < 1$. Now, by Proposition 1.2 (i), we have

$$0 \leq u_\lambda(x) \leq u_{(1-\delta)\lambda^*}(x) \leq F^{-1}\left(\frac{(1-\delta)\lambda^*}{\lambda^*}\|F\|_\infty\right) = F^{-1}\left((1-\delta)\|F\|_\infty\right) = C(\delta, f).$$

as claimed. \square

3. Upper and lower bound for the extremal parameter

In this section, we give another upper and lower bound for the extremal parameter of problem (1.2) which are, in many cases, sharper than those in (1.6). In fact, we prove Theorems 1.3, 1.4 and 1.5. We also give an estimate on L^∞ -bound for the extremal solution of problem (1.2).

Proof of Theorem 1.3. As before let $A = [a_{i,j}]_{i,j}$ which is positive definite symmetric matrix. By a simple computation we have

$$\begin{aligned}
LF(u) &= F''(u) \left(- \sum_{i,j=1}^N a_{i,j}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \right) + F'(u) \left(- \sum_{i,j=1}^N a_{i,j}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + c(x) \cdot \nabla u \right) \\
&= \frac{f'(u)}{f^2(u)} \nabla u A (\nabla u)^T + \frac{Lu}{f(u)} = \lambda = L(\lambda \psi_L).
\end{aligned}$$

It follows that $L(F(u(x)) - \lambda\psi_L(x)) \geq 0$ for all $x \in \Omega$. On the other hand, $F(u(x)) - \lambda\psi_L(x) \geq 0$ on $\partial\Omega$, hence, by the maximum principle we must have $F(u(x)) \geq \lambda\psi_L(x)$ for all $x \in \Omega$, so

$$F^{-1}(\lambda\psi_L(x)) \leq u(x) \quad \text{for all } x \in \Omega.$$

Thus

$$\lambda \leq \frac{F(u(x_0))}{\psi_{L,\Omega}}.$$

In particular, the extremal solution of problem (1.2) satisfies

$$F^{-1}(\lambda^*\psi_L(x)) \leq u^*(x) \quad \text{for all } x \in \Omega.$$

Hence

$$\lambda^* \leq \frac{F(u^*(x_0))}{\psi_{L,\Omega}} \leq \frac{F(a_f)}{\psi_{L,\Omega}}.$$

This completes the proof. \square

Now, we give an estimate on L^∞ -bound for the extremal solution of problem (1.2).

Theorem 3.1. *Extremal solution of problem (1.2) satisfies the following*

$$\|f'(u^*)\|_\infty \geq \inf_{0 < t < a_f} \frac{f(t)}{t}. \quad (3.1)$$

Proof. If u^* is singular, then the result is trivial. So we assume u^* is regular. Let $\eta(x)$ be the positive first eigenfunction with corresponding eigenvalue $\mu_1(L^*, \Omega)$ (see problem (1.3)). Now, since u^* is regular there is some $\phi > 0$ such that

$$\begin{cases} L\phi = \lambda^* f'(u^*)\phi & x \in \Omega, \\ \phi = 0 & x \in \partial\Omega, \end{cases}$$

Multiply this by $\eta(x)$ and integrate by parts to see that

$$\int_{\Omega} (\lambda^* f'(u^*) - \mu_1(L^*, \Omega)) \phi \eta \, dx = 0.$$

Thus there is some $x \in \Omega$ such that

$$\lambda^* f'(u^*(x)) \geq \mu_1(L^*, \Omega).$$

Combining this with inequality (1.6) gives the desired result. \square

Combining Theorem 1.3 and the obtained lower bound in (1.6) we conclude that

$$\frac{1}{\psi_{L,\Omega}} \sup_{0 < t < a_f} \frac{t}{f(t)} \leq \lambda^*(L, \Omega, f) \leq \frac{F(a_f)}{\psi_{L,\Omega}}. \quad (3.2)$$

Theorem 1.4 illustrates the remarkable usefulness of (3.2).

Proof of Theorem 1.4. The proof of this theorem is exactly similar to the proof of Theorem 3.1 in [1]. For the convenience of the reader we mention a brief description of the proof.

Take $f_p(t) := f(t^p)$ for $p \geq 1$. It is easy to see that there exists a unique $t_p > 0$ such that

$$\frac{t_p}{f_p(t_p)} = \sup_{t > 0} \frac{t}{f_p(t)} \quad \text{for all } p \geq 1. \quad (3.3)$$

Then, we can show that $t_p \rightarrow 1$ and $t_p^p \rightarrow 0$ as $p \rightarrow +\infty$. Therefore

$$\lim_{p \rightarrow \infty} \sup_{t > 0} \frac{t}{f_p(t)} = \lim_{p \rightarrow \infty} \frac{t_p}{f(t_p^p)} = \frac{1}{f(0)}. \quad (3.4)$$

On the other hand

$$\lim_{p \rightarrow \infty} \frac{1}{f_p(t)} = \begin{cases} \frac{1}{f(0)} & \text{if } 0 \leq t < 1, \\ 0 & \text{if } t > 1. \end{cases}$$

Taking $\zeta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ with $\zeta(t) = 1/f(0)$ for $t \in [0, 1]$ and $\zeta(t) = 1/f_2(t) = 1/f(t^2)$ for $t \in (1, +\infty)$, then $\zeta \in L^1(\mathbb{R}_+)$ and $1/f_p(t) \leq \zeta(t)$ for $p \geq 2$. Now, by the Lebesgue dominated convergence theorem,

$$\lim_{p \rightarrow \infty} \int_0^\infty \frac{ds}{f_p(s)} = \frac{1}{f(0)}. \quad (3.5)$$

Now, estimate (3.2) guarantees that

$$\frac{1}{\psi_{L,\Omega}} \sup_{t > 0} \frac{t}{f_p(t)} \leq \lambda_p^* \leq \frac{1}{\psi_{L,\Omega}} \int_0^{u_p^*(x_0)} \frac{dt}{f_p(t)} \leq \frac{1}{\psi_{L,\Omega}} \int_0^{+\infty} \frac{dt}{f_p(t)}. \quad (3.6)$$

Taking the limit as p tends to infinity in (3.6) and using (3.4) and (3.5), it follows that

$$\lim_{p \rightarrow \infty} \lambda_p^* = \frac{1}{f(0)\psi_{L,\Omega}} \quad \text{and} \quad \lim_{p \rightarrow \infty} u_p^*(x_0) = +\infty,$$

as claimed. \square

In Theorem 1.5, by the super-solution method (Lemma 2.3) we give a lower bound for the extremal parameter of problem (1.2).

Proof of Theorem 1.5. Take an $\alpha \in (0, \frac{\|F\|_\infty}{\psi_\Omega})$ and define $\bar{u}(x) = F^{-1}(\alpha\psi_L(x))$ for $x \in \Omega$. It is evident that $\bar{u} \in C^2(\Omega) \cap C^1(\partial\Omega)$. We show that \bar{u} is a super-solution of (1.2) for $\lambda = \alpha - \alpha^2\beta(\alpha)$. To do this, we compute $\Delta\bar{u}(x)$. Note that if we take $y = F^{-1}(\alpha t)$, then it is easy to see that $y' = \alpha f(y)$ and $y'' = \alpha^2 f(y)f'(y)$. So

$$\begin{aligned} \Delta\bar{u}(x) &= \left[\alpha^2 f'(\bar{u}) |\nabla\psi_L(x)|^2 - \alpha \right] f(\bar{u}) \\ &\leq \left(\alpha^2 \sup_{x \in \Omega} f' \left(F^{-1}(\alpha\psi_L(x)) \right) |\nabla\psi_L(x)|^2 - \alpha \right) f(\bar{u}) \\ &= - \left(\alpha - \alpha^2\beta(\alpha) \right) f(\bar{u}). \end{aligned}$$

In other words, $\Delta\bar{u}(x) + (\alpha - \alpha^2\beta(\alpha))f(\bar{u}) \leq 0$, and since we have $\bar{u}(x) = 0$, $x \in \partial\Omega$, this shows that \bar{u} is a super-solution of (1.2) for $\lambda = \alpha - \alpha^2\beta(\alpha)$, thus, by Lemma 2.3, problem (1.2) with $\lambda = \alpha - \alpha^2\beta(\alpha)$ has a classical solution and hence

$$\lambda^*(L, \Omega, f) \geq \alpha - \alpha^2\beta(\alpha).$$

Taking the supremum over $\alpha \in (0, \frac{\|F\|_\infty}{\psi_\Omega})$, we obtain (1.10). \square

Combining Theorem 1.5, Theorem 1.3 and the estimates in (1.6), we have

$$\max \left\{ \sup_{0 < \alpha < \frac{\|F\|_\infty}{\psi_{L,\Omega}}} \alpha - \alpha^2\beta(\alpha), \frac{1}{\psi_{L,\Omega}} \sup_{0 < t < a_f} \frac{t}{f(t)} \right\} \leq \lambda^*(L, \Omega, f),$$

and

$$\lambda^*(L, \Omega, f) \leq \min \left\{ \frac{F(a_f)}{\psi_{L,\Omega}}, \mu_1(L^*, \Omega) \sup_{0 < t < a_f} \frac{t}{f(t)} \right\}.$$

where $\beta(\alpha) := \sup_{x \in \Omega} f' \left(F^{-1}(\alpha \psi_L(x)) \right) |\nabla \psi_L(x)|^2$.

In the following two examples, we apply the above results for standard nonlinearities $f(u) = e^u$ (as a regular nonlinearity) and $f(u) = \frac{1}{(1-u)^2}$ (as a singular nonlinearity) on the unit ball $B(0, 1) \subseteq \mathbb{R}^N$.

Example 3.1. *Consider the following problem*

$$\begin{cases} -\Delta u + \frac{-2x \cdot \nabla u}{1 + |x|^2} = \lambda e^u & x \in B(0, 1), \\ u = 0 & x \in \partial B(0, 1). \end{cases} \quad (3.7)$$

Here, we have

$$c(x) = \frac{-2x}{1 + |x|^2}, \quad L = -\Delta + c(x) \cdot \nabla, \quad f(u) = e^u, \quad \Omega = B(0, 1).$$

Now, we look for radial solution for torsion function ψ_L . If there exists smooth function $\varphi : [0, 1] \rightarrow \mathbb{R}$ such that $\psi_L = \varphi(|x|)$, then it is easy to see that φ satisfies the following

$$\begin{cases} \varphi''(|x|) + \left(\frac{N-1}{|x|} + \frac{2|x|}{1 + |x|^2} \right) \varphi'(|x|) = -1 \\ \varphi(1) = 0. \end{cases}$$

Solving the above problem, we get

$$\psi_L(x) = \varphi(|x|) = \frac{N(1 - |x|^2) + 2 \ln \left(\frac{2}{1 + |x|^2} \right)}{2N(N+2)}.$$

Thus

$$\psi_{L,\Omega} = \psi_L(0) = \frac{N + \ln 4}{2N(N+2)}.$$

By (3.2), we have

$$\frac{2N(N+2)}{e(N + \ln 4)} \leq \lambda^*(L, B(0, 1), e^u) \leq \frac{2N(N+2)}{N + \ln 4}. \quad (3.8)$$

One can also apply Theorem 1.5 to obtain another lower bound for the extremal parameter of problem (3.7). Here, we have

$$f'(t) = e^t, \quad F(t) = 1 - e^{-t}, \quad F^{-1}(t) = -\ln(1 - t).$$

Thus

$$\beta(\alpha) = \sup_{x \in B(0,1)} \frac{|\nabla \psi_L(x)|^2}{1 - \alpha \psi_L(x)} = \sup_{0 < t < 1} \frac{\varphi'^2(t)}{1 - \alpha \varphi(t)} \quad \text{for all } 0 < \alpha < \frac{2N(N+2)}{N + \ln 4}.$$

It can be easily checked that

$$\beta(\alpha) = \varphi'^2(1) = \frac{(N+1)^2}{N^2(N+2)^2} \quad \text{for all } 0 < \alpha < \frac{2N^2(N+2)}{(N+1)^2}.$$

On the other hand

$$\sup_{0 < \alpha < \frac{2N(N+2)}{N+\ln 4}} \alpha - \alpha^2 \beta(\alpha) \geq \sup_{0 < \alpha < \frac{2N^2(N+2)}{(N+1)^2}} \alpha - \alpha^2 \beta(\alpha) = \frac{2N^3}{(N+1)^2}.$$

Hence

$$\frac{2N^3}{(N+1)^2} \leq \lambda^*(L, B(0, 1), e^u).$$

Note that this lower bound is better than the one in (3.8) for all $N \geq 3$.

Example 3.2. Consider the following problem

$$\begin{cases} -\Delta u + \frac{-2x \cdot \nabla u}{1+|x|^2} = \frac{\lambda}{(1-u)^2} & x \in B(0, 1), \\ u = 0 & x \in \partial B(0, 1). \end{cases} \quad (3.9)$$

By Example 3.1, we know that

$$\psi_L = \varphi(|x|) = \frac{N(1-|x|^2) + 2 \ln \left(\frac{2}{1+|x|^2} \right)}{2N(N+2)} \quad \text{and} \quad \psi_{L,\Omega} = \psi_L(0) = \frac{N+\ln 4}{2N(N+2)}.$$

By (3.2), we have

$$\frac{8N(N+2)}{27(N+\ln 4)} \leq \lambda^*(L, B(0, 1), (1-u)^{-2}) \leq \frac{2N(N+2)}{3(N+\ln 4)}. \quad (3.10)$$

Again, one can also apply Theorem 1.5 to obtain another lower bound for the extremal parameter of problem (3.9). It can be easily checked that

$$\beta(\alpha) = 2\varphi'^2(1) = \frac{2(N+1)^2}{N^2(N+2)^2} \quad \text{for all } 0 < \alpha < \frac{2N^2(N+2)}{3(N+1)^2}.$$

On the other hand

$$\sup_{0 < \alpha < \frac{2N(N+2)}{3(N+\ln 4)}} \alpha - \alpha^2 \beta(\alpha) \geq \sup_{0 < \alpha < \frac{2N^2(N+2)}{3(N+1)^2}} \alpha - \alpha^2 \beta(\alpha) = \frac{2N^2(3N+2)}{9(N+1)^2}.$$

Hence

$$\frac{2N^2(3N+2)}{9(N+1)^2} \leq \lambda^*(L, B(0, 1), (1-u)^{-2}).$$

Note that this lower bound is better than the one in (3.8) for all $N \geq 2$.

We conclude this section, by proving the last assertion of Proposition 1.2.

Proof of Proposition 1.2 (ii). Set $\lambda(\alpha) = \alpha - \alpha^2 \beta(\alpha)$ for all $0 < \alpha < \frac{\|F\|_\infty}{\psi_{L,\Omega}}$, where

$$\beta(\alpha) := \sup_{x \in \Omega} f' \left(F^{-1}(\alpha \psi_L(x)) \right) |\nabla \psi_L(x)|^2.$$

Clearly $\lambda(\alpha) \rightarrow 0$ as $\alpha \rightarrow 0^+$. By Theorem 1.3 and Theorem 1.5 we have

$$\psi_L(x) \leq \frac{u_{\lambda(\alpha)}(x)}{\lambda(\alpha)} \leq \frac{1}{1-\alpha\beta(\alpha)} \psi_L(x) \quad \text{for all } x \in \overline{\Omega}. \quad (3.11)$$

Taking the limit on both sides of (3.11) as $\alpha \rightarrow 0^+$, we then have the conclusion of Proposition 1.2 (ii). \square

4. Application to the explosion problem in a flow

In this section, we apply previous results to the explosion problem in a flow. First, we determine the behavior of the extremal parameter of problem (1.4) when the flow $c(x)$ is divergence-free (see Theorem 1.1) and then we prove Theorem 1.2.

Proof of Theorem 1.1. By Lemma 2.2 and Theorem 1.3 we have

$$\frac{1}{\psi_{A,\Omega}} \sup_{t>0} \frac{t}{f(t)} \leq \lambda^*(A) \leq \frac{F(a_f)}{\psi_{A,\Omega}}. \quad (4.1)$$

Now, by Theorem A and estimate (4.1) the proof of Theorem 1.1 is complete. \square

Theorem 1.1 completely determine the behaviour of extremal parameter of problem (1.4) when $c(x)$ is divergence-free. But there is still another interesting case when $c(x)$ is not divergence-free. As it is mentioned, in Theorem 1.2, we completely determine the behaviour of extremal parameter of problem (1.4) for a wide class of flows $c(x)$ which are not divergence-free.

Proof of Theorem 1.2. Define

$$\varphi(r) = \int_0^1 \frac{\int_0^t s^{N-1} g^A(s) ds}{t^{N-1} g^A(t)} dt - \int_0^r \frac{\int_0^t s^{N-1} g^A(s) ds}{t^{N-1} g^A(t)} dt \quad \text{for all } 0 \leq r \leq 1,$$

where

$$g(r) := e \int_0^r s \rho(s) ds \quad \text{for } 0 \leq r \leq 1.$$

Then (as it is described in Example 3.1) it is not hard to check that $\psi_{L_A} = \varphi(|x|)$ and since the function

$$r \mapsto \int_0^r \frac{\int_0^t s^{N-1} g^A(s) ds}{t^{N-1} g^A(t)} dt \quad \text{for all } 0 \leq r \leq 1,$$

is increasing, so

$$\psi_{L_{A,B}} = \int_0^1 \frac{\int_0^t s^{N-1} g^A(s) ds}{t^{N-1} g^A(t)} dt. \quad (4.2)$$

Making the change of variable $s = th$ in the interior integral in (4.2), we get

$$\int_0^t s^{N-1} g^A(s) ds = t^N \int_0^1 h^{N-1} g^A(th) dh.$$

Thus

$$\psi_{L_{A,B}} = \int_0^1 t \frac{\int_0^1 h^{N-1} g^A(th) dh}{g^A(t)} dt.$$

- (i) If there exists $x_0 \in [0, 1]$ such that $\rho(x_0) < 0$, then the continuity of ρ implies that there exists an interval $I = [a, b] \subseteq [0, 1]$ such that ρ is negative on I . This means that the function g defined above is strictly decreasing on I . Choose an $\epsilon > 0$ such that

$$0 < \frac{3}{2} - \sqrt{\frac{2a}{b} + \frac{1}{4}} < \epsilon < 1 - \frac{a}{b} < 1. \quad (4.3)$$

It is easy to see that inequality (4.3) implies that

$$(1 - \frac{\epsilon}{2})(1 - \epsilon) < \frac{a}{b} < 1 - \epsilon. \quad (4.4)$$

Now, since the function g is strictly decreasing on I , then for all $A \geq 1$ we have

$$\begin{aligned} \psi_{L_A, B} &= \int_0^1 t \frac{\int_0^1 h^{N-1} g^A(th) dh}{g^A(t)} dt \geq \int_{a/(1-\epsilon)}^b \frac{t \int_{1-\epsilon}^{1-(\epsilon/2)} h^{N-1} g^A(th) dh}{g^A(t)} dt \\ &\geq \int_{a/(1-\epsilon)}^b t \left(\frac{g(t(1 - (\epsilon/2)))}{g(t)} \right)^A dt \int_{1-\epsilon}^{1-(\epsilon/2)} h^{N-1} dh \\ &\geq \left(\frac{g(b(1 - (\epsilon/2)))}{g(a/(1 - \epsilon))} \right)^A \int_{a/(1-\epsilon)}^b t dt \int_{1-\epsilon}^{1-(\epsilon/2)} h^{N-1} dh. \end{aligned} \quad (4.5)$$

By (4.4), we know that $a < b(1 - (\epsilon/2)) < a/(1 - \epsilon) < b$, therefore

$$\frac{g(b(1 - (\epsilon/2)))}{g(a/(1 - \epsilon))} > 1.$$

Thus

$$\left(\frac{g(b(1 - (\epsilon/2)))}{g(a/(1 - \epsilon))} \right)^A \longrightarrow \infty \text{ as } A \longrightarrow \infty.$$

Now (4.5) guarantees that $\psi_{L_A, B} \longrightarrow \infty$ as $A \longrightarrow \infty$.

- (ii) If $\rho \geq 0$ and $\rho \not\equiv 0$ on any interval $I \subseteq [0, 1]$, then g is strictly increasing. Let $0 < \epsilon < 1$ be arbitrary, then

$$\begin{aligned} \int_0^1 h^{N-1} g^A(th) dh &= \int_0^\epsilon h^{N-1} g^A(ht) dh + \int_\epsilon^1 h^{N-1} g^A(ht) dh \\ &\leq \epsilon^N g^A(\epsilon t) + (1 - \epsilon) g^A(t). \end{aligned}$$

It then follows that

$$\psi_{L_A, B} \leq \frac{1 - \epsilon}{2} + \epsilon^N \int_0^1 t \left(\frac{g(\epsilon t)}{g(t)} \right)^A dt.$$

Since g is strictly increasing, it is evident that $t \left(\frac{g(\epsilon t)}{g(t)} \right)^A \longrightarrow 0$ as $A \longrightarrow \infty$ pointwise for all $0 \leq t \leq 1$,

on the other hand $t \left(\frac{g(\epsilon t)}{g(t)\epsilon} \right)^A \leq t \in L^1([0, 1])$ for all $A \geq 0$. Now, Lebesgue dominated convergence theorem implies that

$$\lim_{A \rightarrow \infty} \epsilon^N \int_0^1 t \left(\frac{g(\epsilon t)}{g(t)} \right)^A dt = 0.$$

Thus

$$\limsup_{A \rightarrow \infty} \psi_{L_A, B} \leq \frac{1 - \epsilon}{2}.$$

Letting $\epsilon \longrightarrow 1^-$ in the above inequality, we get $\psi_{L_A, B} \longrightarrow 0$ as $A \longrightarrow \infty$.

- (iii) If $\rho \geq 0$ and $\rho \equiv 0$ on some interval $[a, b] \subseteq [0, 1]$, then g is constant on $[a, b]$. Since the function g is increasing on $[0, 1]$, then

$$\psi_{L_A, B} = \int_0^1 t \frac{\int_0^1 h^{N-1} g^A(th) dh}{g^A(t)} dt \leq \int_0^1 t \frac{g^A(t) \int_0^1 h^{N-1} dh}{g^A(t)} dt = \frac{1}{2N} \quad \text{for all } A \geq 0. \quad (4.6)$$

On the other hand, since g is constant on $[a, b]$ we have

$$\begin{aligned} \psi_{L_A, B} &= \int_0^1 \frac{\int_0^t s^{N-1} g^A(s) ds}{t^{N-1} g^A(t)} dt \geq \int_a^b \frac{\int_a^t s^{N-1} g^A(s) ds}{t^{N-1} g^A(t)} dt \\ &= \int_a^b \frac{\int_a^t s^{N-1} ds}{t^{N-1}} dt \\ &= \frac{1}{N} \int_a^b \frac{t^N - a^N}{t^{N-1}} dt \quad \text{for all } A \geq 0. \end{aligned} \quad (4.7)$$

By (4.6) and (4.7) we conclude that

$$C_{N, \rho} := \frac{1}{N} \int_a^b \frac{t^N - a^N}{t^{N-1}} dt \leq \psi_{L_A, B} \leq \frac{1}{2N} \quad \text{for all } A \geq 0,$$

that completes the proof. □

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